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LETTER TO THE EDITOR

Regular and irregular potential scattering

Bruno Eckhardt and Christof Jung
Fachbereich Physik, Universität Bremen, 2800 Bremen 33, West Germany

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Abstract. We present evidence for irregular behaviour in potential scattering and discuss a possible explanation in terms of unstable periodic orbits.

Trajectories in Hamiltonian systems with bounded energy surface can be identified as regular or irregular, depending on their long-time behaviour (Berry 1978). Regular trajectories are confined to N-dimensional tori (where N stands for the number of degrees of freedom), have discrete spectra and correspond to quasiperiodic motion. One finds them in integrable systems and near elliptic islands. Irregular trajectories densely fill higher than N-dimensional subsets of the energy surface and have continuous spectra. They are typical for stochastic layers. This characterisation of irregular trajectories as well as others in terms of Lyapounov exponents, topological and metric entropy or symbolic dynamics all require that the motion does not becomes less complicated as time goes on.

These requirements are not met by scattering trajectories. For instance, in potential scattering, one finds that asymptotically for large positive and large negative times, the particle does not feel the potential anymore and moves along straight lines. Thus, with only a finite time in the interaction region, the above measures of chaos predict 'simple' motion and one is inclined to dismiss the possibility of chaos in scattering problems (the only reference where we could find this argument in writing is Dragt and Finn (1976), but it appears to be commonplace). However, numerical results to be reported below suggest that it is meaningful to distinguish between regular and irregular behaviour in scattering. The characteristic it has in common with bounded chaos is a sensitive dependence on initial conditions.

The specific system we have studied is given by the Hamiltonian

$$H=\tfrac{1}{2}(p_x^2+p_y^2)+(x^3-3xy^2)\exp[-\tfrac{1}{2}(x^2+y^2)]. \quad (1)$$

The Gaussian factor serves as a convenient cutoff for the potential. Our initial conditions are given by the angle under which the particle approaches the potential, the energy and the initial distance, all kept fixed for all runs, together with the impact parameter b. Figure 1 shows the contour lines of the potential together with the ingoing direction. We monitored the cosine of the angle of deflection (scattering angle) and the time delay as compared to free motion (Narnhofer and Thirring 1981).

We call *regular scattering* the situation usually anticipated in scattering: initially, the particle moves along straight lines, then interacts for a finite time and finally heads

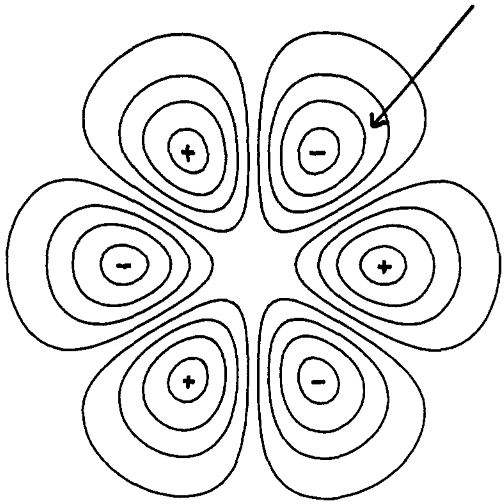


Figure 1. Equipotential contours for the potential in (1). Maxima are marked by +, minima by −. The contours are equally spaced in energy. The arrow marks the ingoing direction for the particle.

off to infinity, again along straight lines. With only a finite interaction time, well known theorems of analysis give a smooth dependence on initial conditions and parameters, which is reflected in smooth variations of, for example, the scattering angle upon changes in the impact parameter. Slightly more complicated is the case where one has asymptotic trapping. Assuming the initial conditions leading to trapping are isolated and well separated, we have regular scattering in intervals in between, perhaps with singularities at the endpoints. This situation is familiar from scattering by radially symmetric potentials, cf Newton (1982) or Goldstein (1980), and will also be considered regular. As a preliminary definition, one might call *irregular scattering* anything more complicated.

An example of what we believe to be evidence for irregular scattering is shown in figure 2. The time delay varies smoothly with impact parameter, except for regions near $b \approx -2.0, -0.7, -0.2, 0.4, 1.9$. Near each small bump, initial conditions can be found such that the time delay diverges. Along with the sharp increase in time delay go strong oscillations in the cosine of the scattering angle. Magnifications by a factor of 400× (figure 3) and 20× (figure 4) reveal a very complicated structure, apparently persisting on ever finer scales. The typical feature of these plots are divergences in time delay and corresponding oscillations in scattering angle for many values of the impact parameter. Since the singularities are so dense, small changes in initial state lead to large changes in the final state, a behaviour known as sensitive dependence on initial conditions for irregular bound trajectories.

A qualitative explanation of this behaviour can be given in terms of unstable periodic orbits. In the (regular) case of orbiting scattering in radially symmetric potentials, one has an unstable periodic orbit at the top of the barrier and the asymptotically trapped orbits spiral towards it. In mathematical language, the particle moves along the stable manifold of the periodic orbit. Our irregular scattering can

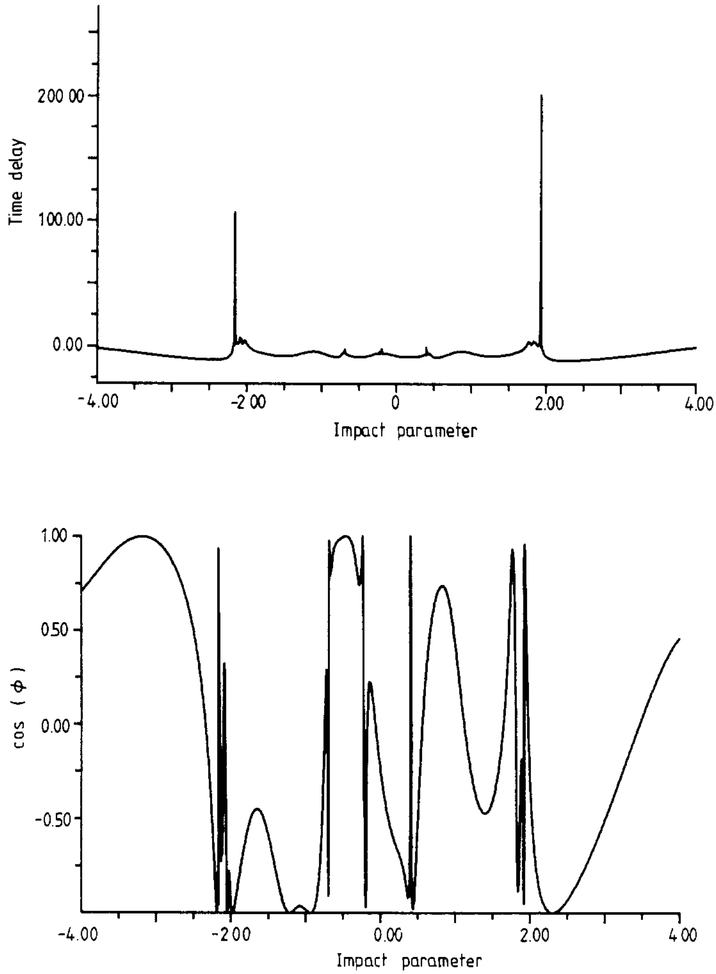


Figure 2. Time delay and cosine of the scattering angle as a function of impact parameter in the interval $b \in [-4.0, 4.0]$.

now be explained as being due to an infinity of periodic orbits, each with a stable manifold extending to infinity, i.e. to the set of ingoing asymptotic states. Whenever we cross one of these manifolds, we find a divergence in time delay and oscillations in the scattering angle.

When do periodic orbits exist? We believe that the methods of Churchill *et al* (1979), who prove existence of unstable periodic orbits in the Hénon–Heiles potential above the dissociation threshold, can immediately be applied to our potential. The dissociation energy of the bound states is $E = 0$, which coincides with the minimal energy of scattering states. We also have an upper limit due to the finite height of the potential barriers in our example. We conclude that irregular scattering can only be observed in an energy interval between the dissociation energy of the bound states and the maximal height of the potential barrier.

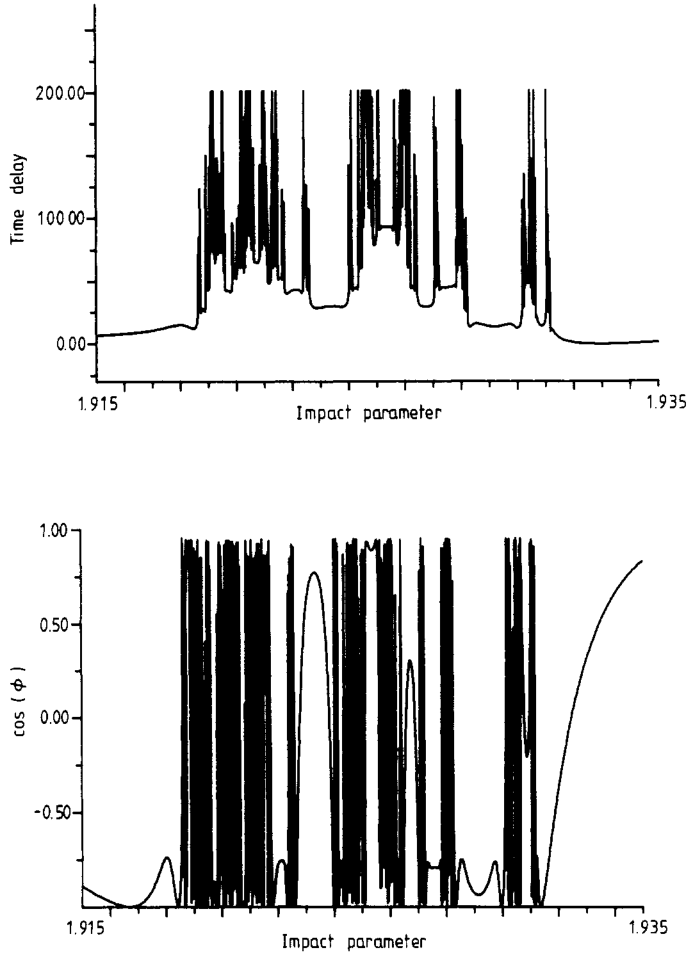
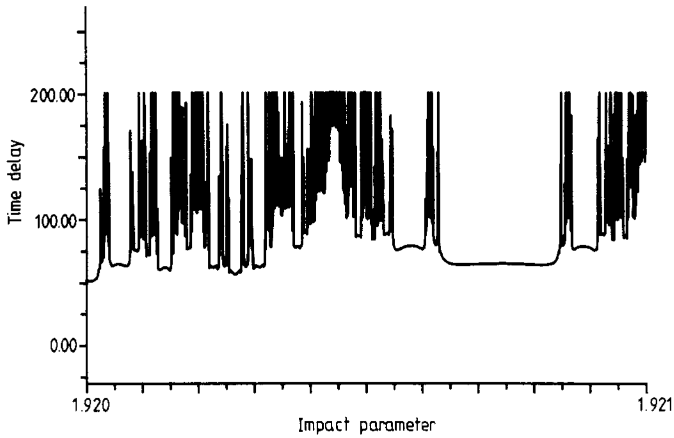


Figure 3. Same as figure 2, except $b \in [1.915, 1.935]$. Trajectories were integrated up to a time delay $\leqslant 200$.

In conclusion, we have presented evidence for irregular behaviour in scattering, the characteristic being a sensitive dependence of scattering data on initial state. We have proposed an explanation in terms of trapping in unstable periodic orbits. Similar behaviour has been found in other potentials and in scattering in a system in 2D hydrodynamics (Manakov and Shchur 1983, Eckhardt and Aref 1986). The extent to which this behaviour is analogous to bounded chaos, its relation to hyperbolic points and homoclinic oscillations and connections to other scattering problems (Dragt and Finn 1976, Moser 1973) is currently under investigation. We expect the Poincaré map for scattering states introduced by Jung (1986) to be a useful mathematical tool.

We would like to thank Professors H Aref and P H Richter, and Dr H J Scholz for valuable discussions.



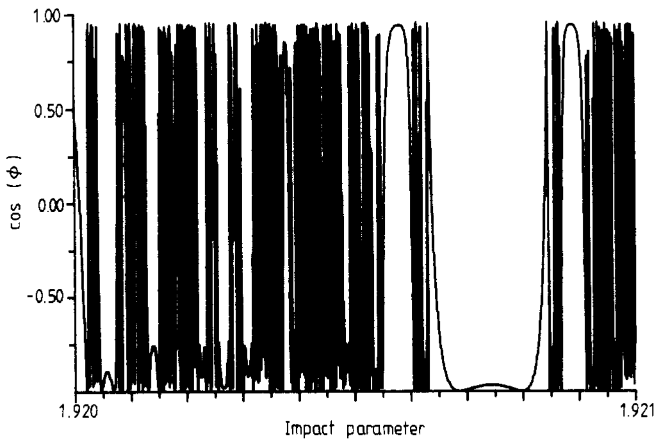


Figure 4. Same as figure 2, except $b \in [1.920, 1.921]$. Trajectories were integrated up to a time delay $\leqslant 200$.

References

Berry M V 1978 *Topics in Non-linear Dynamics* (*AIP Conf. Proc.* **46**) ed S Jorna (New York: AIP) pp 16-120

Churchill R C, Pecelli G and Rod D L 1979 *Stochastic Behavior in Classical and Quantum Hamiltonian Systems* (*Lecture Notes in Physics* **93**) ed G Casati and J Ford (Berlin: Springer) pp 76-136

Dragt A J and Finn J H 1976 *J. Geophys. Res.* **81** 2327-40

Eckhardt B and Aref H 1986 in preparation

Goldstein H 1980 *Classical Mechanics* (Reading, MA: Addison-Wesley) 2nd edn

Jung C 1986 *J. Phys. A: Math. Gen* **19** 1345-53

Manakov S V and Shchur L N 1983 *Zh. Eksp. Teor. Fiz. Pis. Red.* **37** 45-8 (*JETP Lett.* **37** 54-7)

Moser J 1973 *Stable and Random Motions* (Princeton, NJ: Princeton University Press)

Narnhofer H and Thirring W 1981 *Phys. Rev.* A **23** 1688-97

Newton R G 1982 *Scattering Theory of Waves and Particles* (Berlin: Springer)